A closed-form solution in a dynamical system related to Bianchi IX

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Abstract

The Bianchi IX cosmological model in vacuum can be represented by several six-dimensional dynamical systems. In one of them we present a new closed form solution expressed by a third Painlevé function.

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The Bianchi IX cosmological model in vacuum can be defined by the metric [4]

$$\mathrm{d}s^2 = \sigma^2 \mathrm{d}t^2 - \gamma_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta, \tag{1}$$

$$\gamma_{\alpha\beta} = \eta_{ab} e^a_\alpha e^b_\beta,\ \eta = \mathrm{diag}(A, B, C), \tag{2}$$

in which e^a_α are the components of the three frame vectors, and $\sigma^2 = \pm 1$ according as the metric is Minkovskian or Euclidean. Introducing the logarithmic time τ by the hodograph transformation

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{\sqrt{ABC}}, \tag{3}$$

this gives rise to the six-dimensional system of three second order ODEs

$$\sigma^2 (\log A)'' = A^2 - (B - C)^2 \text{ and cyclically, } ' = \mathrm{d}/\mathrm{d}\tau, \tag{4}$$

or equivalently

$$\sigma^2 (\log \omega_1)'' = \omega_2^2 + \omega_3^2 - \omega_2^2 \omega_3^2/\omega_1^2 \text{ and cyclically,} \tag{5}$$

under the change of variables

$$A = \omega_2\omega_3/\omega_1,\ \omega_1^2 = BC \text{ and cyclically.} \tag{6}$$

If one introduces the six variables

$$y_1 = \frac{A}{\sigma},\ z_1 = \frac{\mathrm{d}}{2\mathrm{d}\tau} \log(BC), \quad \text{and cyclically,} \tag{7}$$

the dynamical system (4) can be alternatively represented by [2]

$$\frac{\mathrm{d}y_j}{\mathrm{d}\tau} = -y_j (z_j - z_k - z_l),\ \frac{\mathrm{d}z_j}{\mathrm{d}\tau} = -y_j (y_j - y_k - y_l), \tag{8}$$

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in which (j,k,l) is any permutation of $(1,2,3)$. This system admits the first integral

$$\sigma^{-2}K_1 = y_1^2 + y_2^2 + y_3^2 - 2y_2y_3 - 2y_3y_1 - 2y_1y_2 - (z_1^2 + z_2^2 + z_3^2 - 2z_2z_3 - 2z_3z_1 - 2z_1z_2). \quad (9)$$

All the single valued solutions of (4) are known in closed form [1, 8], except a four-parameter solution [5] which would extrapolate the three-parameter elliptic solution [1]

$$\omega_j = \sigma\sqrt{\wp(\tau-\tau_0, g_2, g_3) - e_j},\ j = 1,2,3,\ K_1 = 0, \quad (10)$$

in which $\wp, g_2, g_3, e_j$ is the classical notation of Weierstrass,

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),\ (g_2, g_3, e_j) \text{ complex}, \quad (11)$$

and g_2, g_3, τ_0 are arbitrary. The solution (10) also represents the motion of a rigid body around its center of mass (Euler, 1750),

$$\sigma\omega_1' = \omega_2\omega_3,\ \text{and cyclically}. \quad (12)$$

Any hint to find the above mentioned missing four-parameter solution would be welcome, and some indications can be found in Ref. [5]. In the present Letter, we present such a hint, as a five-parameter solution of (8). Despite its lack of physical meaning, it could share some analytic structure with the unknown solution and therefore provide a useful insight.

When one coordinate y_i vanishes, say $y_1 = 0$, the correspondence (3) between the physical time t and the logarithmic time τ breaks down, but the system (8), whose investigation was then started in Ref. [6], can be integrated in closed form.

Taking account of the two additional first integrals [6],

$$c = -z_1,\ K_2 = y_2y_3e^{-2c\tau}, \quad (13)$$

the system reduces to

$$\left\{\begin{array}{l} (z_2 - z_3)' = -(y_2 + y_3)(y_2 - y_3), \\ (y_2 + y_3)' = -(y_2 - y_3)(z_2 - z_3) - c(y_2 + y_3), \\ (y_2 - y_3)' = -(z_2 - z_3)(y_2 + y_3) - c(y_2 - y_3), \\ (z_2 + z_3)' = -(y_2 - y_3)^2. \end{array}\right. \quad (14)$$

For $c = 0$, the system for $z_2 - z_3, y_2 + y_3, y_2 - y_3$ is another Euler top, whose general solution is

$$\left\{\begin{array}{l} y_1 = 0,\ z_1 = 0, \\ z_2 - z_3 = \sqrt{\wp(\tau-\tau_1, g_2, g_3) - e_1}, \\ y_2 + y_3 = \sqrt{\wp(\tau-\tau_1, g_2, g_3) - e_2}, \\ y_2 - y_3 = \sqrt{\wp(\tau-\tau_1, g_2, g_3) - e_3}, \\ z_2 + z_3 = \zeta(\tau-\tau_1, g_2, g_3) + e_3(\tau-\tau_1) + 2z_0, \end{array}\right. \quad (15)$$

in which g_2, g_3, τ_1, z_0 are the four arbitrary constants.

For $c \neq 0$, the elimination of (y_3, z_2, z_3) between the two first integrals and the original system yields the general solution

$$\left\{\begin{array}{l} y_1 = 0,\ z_1 = -c \neq 0, \\ -y_j'' + \dfrac{{y_j'}^2}{y_j} + y_j^3 - K_2^2e^{-4c\tau}y_j^{-1} = 0,\ j = 2 \text{ or } 3, \\ y_2y_3 = K_2e^{2c\tau}, \\ z_2 - z_3 = \dfrac{y_3'}{2y_3} - \dfrac{y_2'}{2y_2}, \\ z_2 + z_3 = \dfrac{(y_2 - y_3)^2 - (z_2 - z_3)^2 + \sigma^{-2}K_1 - c^2}{2c}, \end{array}\right. \quad (16)$$

and the second order ordinary differential equation for y_2 (or for y_3 as well) is a third Painlevé equation [7], with the correspondence

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)^2 - \frac{\mathrm{d}w}{\xi \mathrm{d}\xi} + \frac{\alpha w^2 + \gamma w^3}{4\xi^2} + \frac{\beta}{4\xi} + \frac{\delta}{4w}, \tag{17}$$

$$w = y_2 \text{ or } y_3,\ \xi = e^{-2c\tau},\ \alpha = 0,\ \beta = 0,\ \gamma = 1,\ \delta = -K_2^2 c^{-4}. \tag{18}$$

In the generic case $cK_2 \neq 0$, this solution is a meromorphic function of τ, with a transcendental dependence on the two constants of integration other than (c, K_1, K_2).

What is remarkable is that the unknown four-parameter solution of (4) and the Painlevé III solution (16) of (8) are both extrapolations of an Euler top. This suggests looking for another possible three-dimensional Euler top in the six-dimensional physical system (4). Such a three-dimensional subsystem would necessarily correspond to a non self-dual curvature [3].

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